KD

CHELTENHAM GIRLS HIGH SCHOOL

YEAR 12
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATIONS
August 2009

EXTENSION 1 MATHEMATICS

Time allowed : 2 hours (plus 5minutes reading time).

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- All necessary working should be shown in every question.
- Marks may be deducted for careless or badly arranged work
- Answers are to be completed in blue or black pen on supplied writing paper.
- **Write your name or number** on the top right of each page.
- Board approved calculators may be used.
- Diagrams are not drawn to scale.

Name : ______________________________

Student number :______________________

Class Teacher :_____________________

Question	**1**	**2**	**3**	**4**	**5**	**6**	**7**	Total	%
Mark	/12	/12	/12	/12	/12	/12	/12	/84	

Question 1 **Start a new page.** **Marks**

a) Differentiate $y = \cos^{-1} 3x$ 2

b) Find $\int \cos^2 x \, dx$ 2

c) Show that $F(x) = \dfrac{\sqrt{1-x^2}}{\sin^{-1} x}$ is an odd function. 2

d) Find the coordinates of the point P which divides the interval AB with end points A(2, 3) and B(7, -7) externally in the ratio 4:9. 3

e) Evaluate $\int_1^{\sqrt{3}} \dfrac{x}{x^2+1} \, dx$, giving your answers in simplified, exact form. 3

Question 2 **Start a new page.** **Marks**

a) i) Factorise $a^3 - b^3$. 1

ii) Hence or otherwise show that if $\sin A \neq \cos A$ then $\dfrac{2\sin^3 A - 2\cos^3 A}{\sin A - \cos A} = 2 + \sin 2A$. 3

b) Solve the inequality $\dfrac{2x-3}{x+2} \geq 3$ 3

c) Use calculus to find the value of x which will maximise the value of $3\sin x + 2\cos x$. Give your answer in radians, correct to 2 decimal places. (You need not find the maximum value or prove that it will actually be a maximum.) 2

d) Use $x = \log u$ to find $\int \dfrac{e^x}{\sqrt{1-e^{2x}}} \, dx$. 3

Question 3 is on the next page.

Question 3 **Start a new page.** **Marks**

a) A polynomial is given by $f(x) = x^3 + ax^2 + 7x + b$. Find the values of a and b if $(x+3)$ is a factor of $f(x)$ and when $f(x)$ is divided by $(x-2)$ the remainder is 35. 4

b) The equation $\ln x + x = 2$ has only one root.

i) Show that the root lies between $x = 1$ and $x = 2$. 1

ii) With 2 as a first estimate of the root, use one application of Newton's method to find an other approximation to the root, correct to 2 decimal places. 3

c) Use the method of mathematical induction to prove that $9^{n+2} - 4^n$ is divisible by 5 for all positive integers n. 4

Question 4 **Start a new page.** **Marks**

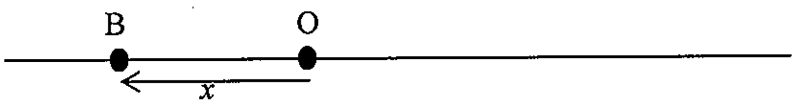
a) Let $f(x) = 2x - x^2$ for $x \le 1$. This function has an inverse $f^{-1}(x)$.

i) Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same diagram. 2

ii) Find an expression for $f^{-1}(x)$. 3

iii) Evaluate $f^{-1}\left(\dfrac{3}{4}\right)$. 2

b) The point B is allowed to move along a fixed line such that its acceleration in *cm/s*^{2} is given by $\ddot{x} = x^2 + 3$ where x is the displacement of OB.



Initially B is at rest, 3*cm* to the left if O (where $x = -3$).

i) Prove that B will move with velocity v such that $v^2 = \dfrac{2}{3}x^3 + 6x + 36$. 2

ii) By the time that B is 3*cm* to the right of O, how fast is it moving, to the nearest 0.01*cm/s*? 2

iii) Explain how we know that B will <u>never</u> move towards the left. 1

Question 5 **Start a new page.** **Marks**

a) Mr and Mrs Bunn, Mr and Mrs Dunn, Mr and Mrs Funn, Mr and Mrs Gunn and Mr and Mrs Hunn are a group of 10 friends at a fund-raising dance in aid of the bush fire victims.
How many ways can :

i) they stand in line to do the Macarena? (There are no restrictions on who stands beside whom.) 1

ii) they sit around a round table if no men sit together? 2

iii) 9 of them be chosen to dance a particular dance which requires exactly 9 people? 1

b) A particle is moving in simple harmonic motion on a line. Its maximum acceleration is $2m/s^2$ and its maximum speed is 6 m/s. Find the amplitude and period of its motion. 2

c) When a stone is projected from the origin with a velocity V, directed α degrees above horizontal, the equations of motion horizontally and vertically are given by

$x = Vt\cos\alpha$ and $y = Vt\sin\alpha - 5t^2$. Do NOT prove these results.

i) Show that the maximum height of its flight path is $\dfrac{V^2\sin^2\alpha}{20}$. 2

ii) Prove that the stone will land furthest from the origin when $\alpha = 45^o$. 3

iii) In terms of V, what is this maximum range of the stone? 1

Question 6 **Start a new page.** **Marks**

a) Sketch $y = \dfrac{x}{x^2-4}$, clearly indicating its asymptotes. 4

b) The line $y = x + b$ intersects $x^2 = 12y$ at $P\ (x_1, y_1)$ and $Q\ (x_2, y_2)$.
M is the midpoint of interval PQ.

i) Show this information on a neat sketch. 1

ii) Find the equation of the locus of M. 5

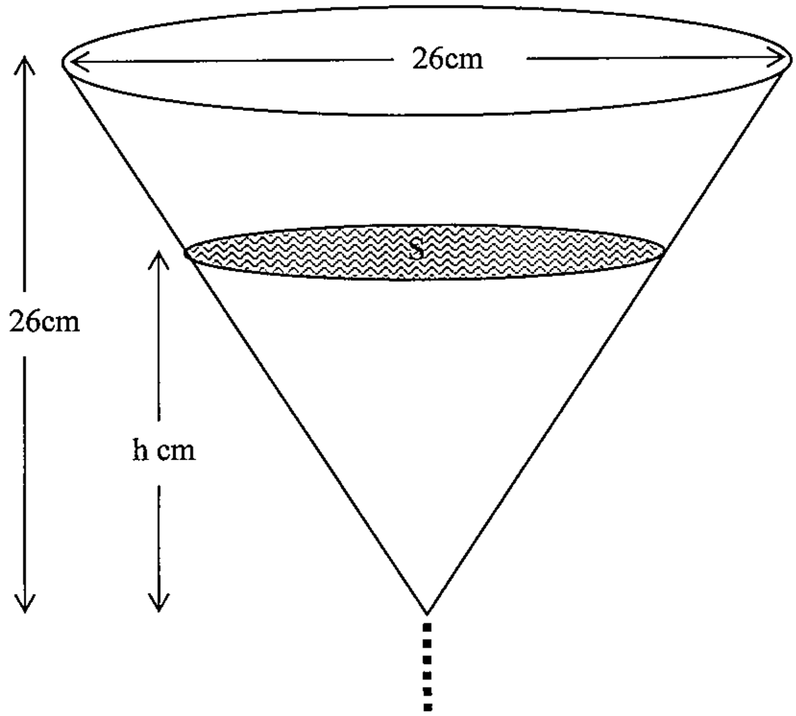
iii) Find the values of b for which P and Q are 2 distinct points. 2

Question 7 **Start a new page.** **Marks**

a) i) Use the result for $\cos(A+B)$ to prove that $\cos 2x = 1 - 2\sin^2 x$. 1

ii) Use the result $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$ to find $\lim\limits_{x \to 0} \dfrac{1-\cos 2x}{x^2}$. 2

b) A conical vessel has height 26*cm* and diameter 26*cm*. Water is leaking out of the bottom at a rate of 80cm^3 per minute. After t minutes the depth of water is *h cm*.



i) Find the surface area S (shaded) of the water in terms of h. 1

ii) Show that the volume of water is given by $V = \frac{1}{12}\pi h^3$. 1

iii) Find expressions for $\dfrac{dV}{dh}$, $\dfrac{dS}{dh}$ and $\dfrac{dV}{dt}$. 3

iv) Find the rate at which the surface area S is decreasing when the water is 10*cm* deep. 4

End of test.

① a) $y' = \frac{-1}{\sqrt{1-9x^2}} \cdot 3$

$= \frac{-3}{\sqrt{1-9x^2}}$

b) $\int \cos^2 x \, dx = \frac{1}{2} \int (1 + \cos 2x) \, dx$

$= \frac{1}{2} \left[x + \frac{1}{2} \sin 2x \right] + c$

c) $F(-x) = \frac{\sqrt{1-(-x)^2}}{\sin^{-1}(-x)}$

$= \frac{\sqrt{1-x^2}}{-\sin^{-1} x}$

$= -F(x)$

$\therefore F(x)$ is odd.

d) $A(2, 3)$ $\quad -4:9 \quad$ $B(7, -7)$

$P = \left(\frac{9 \times 2 - 4 \times 7}{9-4}, \frac{9 \times 3 - 4 \times -7}{9-4} \right)$

$= \left(\frac{18-28}{5}, \frac{27+28}{5} \right)$

$= (-2, 11)$

e) $\frac{1}{2} \int_1^{\sqrt{3}} \frac{2x}{x^2+1} \, dx = \frac{1}{2} \left[\ln(x^2+1) \right]_1^{\sqrt{3}}$

$= \frac{1}{2} \ln 4 - \frac{1}{2} \ln 2$

$= \frac{1}{2} \ln 2$

2) a) i) $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

ii) $\dfrac{2(\sin^3 A - \cos^3 A)}{\sin A - \cos A} = \dfrac{2\cancel{(\sin A - \cos A)}(\sin^2 A + \sin A \cos A + \cos A)}{\cancel{\sin A - \cos A}}$

$= 2(1 + \sin A \cos A)$

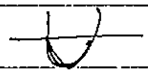
$= 2 + \sin 2A$

b) $\therefore (2x-3)(x+2) \geq 3(x+2)^2$

$2x^2 + x - 6 \geq 3x^2 + 12x + 12$

$x^2 + 11x + 18 \leq 0$

$(x+9)(x+2) \leq 0$



$-9 \leq x \leq -2$

But denominator $\neq 0$

$\therefore x \neq -2$

$\therefore -9 \leq x < -2$

c) $y = 3\sin x + 2\cos x$

$\therefore y' = 3\cos x - 2\sin x$

For max, $y' = 0$

$\therefore 3\cos x = 2\sin x$

$\therefore \dfrac{3}{2} = \dfrac{\sin x}{\cos x}$

$= \tan x$

$\therefore x = 0 \cdot 98279\ldots$

$\doteqdot 0 \cdot 98$

d) $x = \log u \Rightarrow u = e^x$

$\therefore dx = \frac{1}{u}\,du$

$\therefore \displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int \frac{\cancel{u}}{\sqrt{1-u^2}} \cdot \frac{1}{\cancel{u}}\,du$

$= \sin^{-1} u + c$

$= \sin^{-1} e^x + c$

③ a) $f(x) = x^3 + ax^2 + 7x + b$

$f(-3) = 0$

$\therefore \; 0 = -27 + 9a - 21 + b$

$0 = 9a + b - 48 \ldots (1)$

$f(2) = 35$

$\therefore \; 35 = 8 + 4a + 14 + b$

$0 = 4a + b - 13 \ldots (2)$

$(1)-(2)$: $\quad 0 = 5a - 35$

$a = 7$

(1): $\therefore \; 0 = 63 + b - 48$

$\left.\begin{array}{l} b = -15 \\ a = 7 \end{array}\right\}$

b) i) Let $\ln x + x - 2 = f(x)$

$f(1) = \ln 1 + 1 - 2$

$= -1$

< 0

$f(2) = \ln 2 + 2 - 2$

$= \ln 2$

> 0

$\therefore$ There is a root between $1 + 2$.

ii) $f'(x) = \frac{1}{x} + 1$

If $x_1 = 2$, $\quad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 2 - \frac{\ln 2 + 2 - 2}{\frac{1}{2} + 1}$

$= 2 - \frac{2 \ln 2}{3}$

$= 1.53790188$

$\doteqdot 1.54$

3 c) RTP: $9^{n+2} - 4^n = 5A_i$ (A_i integral)

Try $n=1$: $LHS = 9^3 - 4^1$

$= 725$

$= 5 \times 145$

$\therefore$ True for $n=1$

Assume true for $n = k$ (k integral)

ie. $9^{k+2} - 4^k = 5A_k$ ... (1)

Try for $n = k+1$:

$$9^{k+3} - 4^{k+1} = 9 \times 9^{k+2} - 4 \times 4^k$$
$$= 9(5A_k + 4^k) - 4 \times 4^k$$
$$= 45A_k + 9 \times 4^k - 4 \times 4^k$$
$$= 5(9A_k + 4^k)$$

$\therefore$ If true for $n=k$ then true for $n=k+1$

But true for $n=1$

$\therefore$ " " $n=2$

$\therefore$ " " $n=3$

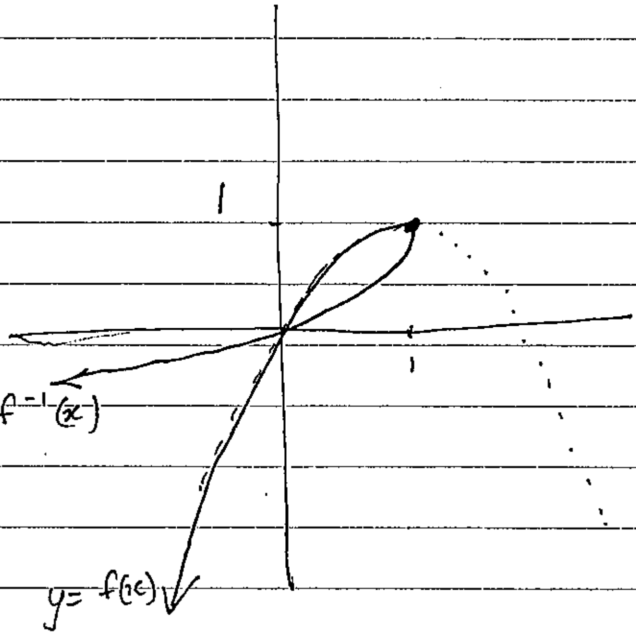
etc.

$\therefore$ True for all positive integers.

④ a) i)

$f(x) = x(2-x)$

$f(1) = 2 - 1 = 1$



ii) $x = 2y - y^2$

$\therefore \quad y^2 - 2y + x = 0$

$$y = \frac{2 \pm \sqrt{4 - 4x}}{2 \times 1}$$

$$= 1 \pm \sqrt{1-x}$$

Inverse is $f^{-1}(x) = 1 - \sqrt{1-x}$

iii) $f^{-1}\left(\frac{3}{4}\right) = 1 - \sqrt{1 - \frac{3}{4}}$

$= 1 - \frac{1}{2}$

$= \frac{1}{2}$

b) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = x^2 + 3$

$\therefore \quad \frac{1}{2}v^2 = \frac{1}{3}x^3 + 3x + c$

When $x = -3$, $v = 0$

$\therefore \quad 0 = -9 - 9 + c$

$c = 18$

$\therefore \quad v^2 = \frac{2}{3}x^3 + 6x + 36$ QED.

ii) When $x = 3$, $v^2 = \frac{2}{3}(3)^3 + 6(3) + 36$

$= 72$

$\therefore \quad v = 8.48528137\ldots$

$\doteqdot 8.49$ cm/s.

iii) $\ddot{x} = x^2 + 3$ is always positive and B is initially at rest. $\therefore$ B will move to the right, getting faster always and thus will <u>never</u> move left.

5) a) iii) n(one left out) = 10

i) n(10 in line) = 10!

(= 3 628 800)

ii) n(alternate around table)

= n(men around table) × n(women between them)

= 4! × 5!

(= 2880)

[Award ~~one mark for using~~ 9!, 4! or 5! in working.]

[or one mark for indicating that they need to alternate.]

b) Let $x = A \sin nt$ for SHM

$\therefore \dot{x} = An \cos nt$, which has maximum of An

$\therefore \ddot{x} = -An^2 \sin nt$, " " " " An^2

$\therefore$ $An^2 = 2$ ... ①

$An = 6$ ... ②

$n = \frac{1}{3}$

②: $\frac{1}{3}A = 6$

$\therefore A = 18$

$T = \frac{2\pi}{n}$

$= \frac{2\pi}{\frac{1}{3}}$

$= 6\pi$

$\therefore$ Amplitude = 18 cm, period = 6π seconds.

c) i) $y = Vt \sin\alpha - 5t^2$

$\therefore \dot{y} = V \sin\alpha - 10t$

For max y, $\dot{y} = 0$ $\therefore V \sin\alpha = 10t$

$t = \frac{V\sin\alpha}{10}$

$\therefore$ Max $y = V\left(\frac{V\sin\alpha}{10}\right)\sin\alpha - 5\left(\frac{V\sin\alpha}{10}\right)^2$

$= \frac{V^2\sin^2\alpha}{10} - \frac{5V^2\sin^2\alpha}{100}$

$= \frac{V^2\sin^2\alpha}{20}$ QED.

(5) c ii) When $y=0$, $Vt\sin\alpha - 5t^2 = 0$

$$t(V\sin\alpha - 5t) = 0$$

$$\therefore t = 0 \quad \text{or} \quad t = \frac{V\sin\alpha}{5}$$

When $t = \frac{V\sin\alpha}{5}$, $x = \frac{V^2\sin\alpha\cos\alpha}{5}$

$$= \frac{V^2\sin 2\alpha}{10} \quad \ldots (1)$$

which has maximum when $\sin 2\alpha = 1$

$$\therefore 2\alpha = 90^\circ$$

$$\therefore \alpha = 45^\circ$$

Q.E.D.

iii) ~~Subst. $\alpha = 45^\circ$ into range: $x = V$~~

From (ii), maximum range $= \frac{V^2}{10}$.

⑥ a) For vertical asymptotes, $x^2 - 4 = 0$

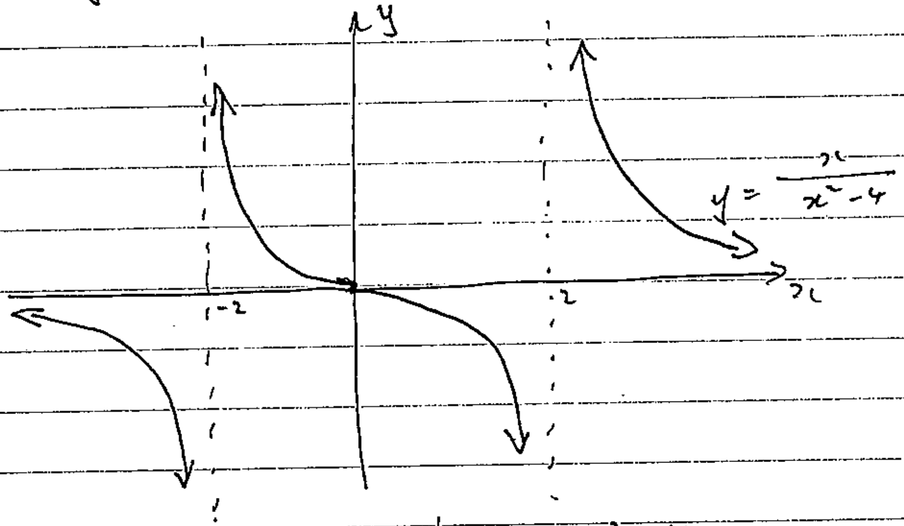
$\therefore \quad x = \pm 2$ #

$$\lim_{x \to \pm\infty} \frac{x}{x^2-4} = \lim_{x \to \pm\infty} \frac{\frac{1}{x}}{1 - \frac{4}{x^2}}$$

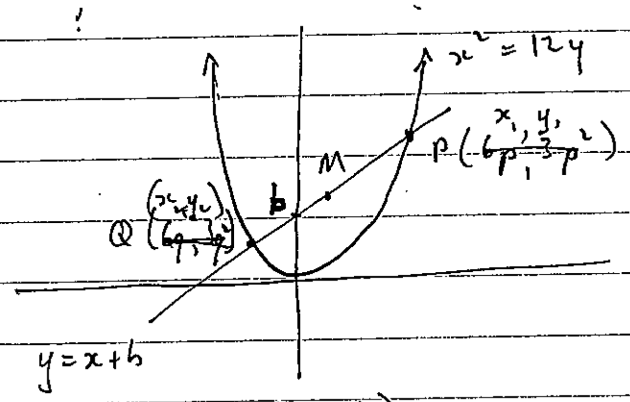
$$= \frac{0}{1-0}$$

$$= 0$$

$\therefore$ $y = 0$ is horizontal asymptote. #



b) i)



ii) $M = \cancel{\left(\frac{6p+6q}{2}, \frac{3p^2+3q^2}{2}\right)}$

$$\left.\begin{aligned} x^2 &= 12y \\ y &= x + b \end{aligned}\right\}$$

$\therefore \quad x^2 = 12x + 12b \quad \ldots \text{①}$

$x^2 - 12x - 12b = 0$

$\therefore \quad x = \frac{12 \pm \sqrt{144 + 48b}}{2}$

$= 6 \pm \sqrt{36 + 12b} \equiv x_1, x_2$

$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

$= \left(\frac{6 + \sqrt{\cdots} + 6 - \sqrt{\cdots}}{2}, \frac{6 + \sqrt{\cdots} + b + 6 - \sqrt{\cdots} + b}{2}\right)$

$= (6, 6 + b)$

$\therefore$ Locus of M is $x = 6$

iii) For distinct points, in ①, $\Delta > 0$

$\therefore 144 + 48b > 0$

$48b > -144$

$b > -3$

⑦ a) i) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\therefore \cos 2x = \cos x \cos x - \sin x \sin x$

$= \cos^2 x - \sin^2 x$

$= 1 - 2\sin^2 x \qquad (\because \sin^2 x + \cos^2 x = 1)$

ii) $\lim_{x \to 0} \frac{1 - \cos 2x}{x^2} = \lim_{x \to 0} \frac{1 - (1 - 2\sin^2 x)}{x^2}$

$= \lim_{x \to 0} \frac{2\sin^2 x}{x^2}$

$= 2 \times 1^2$

$= 2$

b) i) Diam. $= h$

$\therefore$ rad. $= \frac{h}{2}$

$\therefore S = \pi r^2$

$= \pi \left(\frac{h}{2}\right)^2$

$= \frac{\pi h^2}{4}$

ii) $V = \frac{1}{3} S h$

$= \frac{1}{3} \times \frac{\pi h^2}{4} \times h$

$= \frac{1}{12} \pi h^3$ QED

iii) $\frac{dV}{dh} = \frac{1}{4} \pi h^2$

$\frac{dS}{dh} = \frac{1}{2} \pi h$

$\frac{dV}{dt} = -80$ [accept $+80$]

iv) $\frac{dS}{dt} = \frac{dS}{dh} \cdot \frac{dh}{dV} \cdot \frac{dV}{dt}$

$= \frac{\pi h}{2} \cdot \frac{4}{\pi h^2} \cdot \frac{-80}{1}$

$= \frac{-160 \pi h}{\pi h^2}$

When $h = 10$, $\frac{dS}{dt} = \frac{-160}{10}$

$= -16$

ie. surface decreasing by $16\ cm^2/min$.